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2000 J. Phys. A: Math. Gen. 33 2059

(http://iopscience.iop.org/0305-4470/33/10/308)

J. Phys. A: Math. Gen. **33** (2000) 2059–2068. Printed in the UK PII: S0305-4470(00)07667-8

Recursive construction of the generator for Lagrangian gauge symmetries

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Received 9 September 1999, in final form 5 January 2000

Abstract. We obtain, for a subclass of structure functions characterizing a first-class Hamiltonian system, recursive relations from which the general form of the local symmetry transformations can be constructed in terms of the independent gauge parameters. We apply this to a non-trivial Hamiltonian system involving two primary constraints, as well as two secondary constraints of the Nambu–Goto type. We also illustrate for a pure Chern–Simons theory how this formalism can be extended to a system with first- and second-class constraints.

1. Introduction

The problem of finding the most general local symmetries of a Lagrangian has been pursued by various authors, using either Lagrangian [1–4] or Hamiltonian techniques [5–8]. It is well known since the work of Dirac [9] that the generator G of the local symmetries of a Hamiltonian system is given as a linear combination of the first-class constraints of that system. However, in order to also be the generator of local symmetries of the action, restrictions must be imposed on the gauge functions parametrizing the generator G. Indeed, the number of independent gauge parameters characterizing a Lagrangian symmetry is equal to the number of first-class *primary* constraints, that is, constraints which follow solely from the definition of the canonical momenta. Hence, if there exist first-class *secondary* constraints generated by the usual Dirac algorithm from the primary ones by the requirement of persistence in time of the constraints, such restrictions must exist.

In a recent paper [10] we showed that the requirement of commutativity of the time derivative operation with an infinitesimal gauge variation generated by G was the only input needed to obtain the restrictions on the gauge parameters determining the most general form of the generator of Lagrangian symmetries. The analysis was performed entirely within the Hamiltonian framework. On the basis of the above commutativity requirement we subsequently derived [11] a simple differential equation for the generator encoding, in particular, the restrictions on the gauge parameters in the form of first-order coupled differential equations. This coupled set of equations can, in general, not be solved for the gauge parameters. Nevertheless, for a subclass of the structure functions, characteristic of a number of physically interesting systems (which include Yang–Mills theories [11]), an explicit solution can be

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obtained. Though the solution in this case has been constructed previously [13], we believe our procedure and presentation to be considerably simpler.

This paper is organized as follows. In section 2 we derive and explicitly solve for a subclass of structure functions the recursive relations for the gauge parameters, and thereby obtain the explicit form of the corresponding gauge variations at the Lagrangian level. In section 3 we then illustrate this general scheme in terms of two examples: we first consider a (purely first-class) non-trivial model discussed in the literature [12], whose secondary constraints are identical with the primary constraints of the Nambu–Goto model. Our result for the gauge transformation is found to agree with that quoted in the literature. Following this, we then illustrate for the case of a pure Abelian Chern–Simons theory, two possible procedures for dealing with a system having first- and second-class constraints.

2. Recursive construction of a gauge generator

Consider a Hamiltonian system whose dynamics is described by the total Hamiltonian

$$H_T = H_c + \sum_a v^a \Phi_a \tag{1}$$

where H_c is the canonical Hamiltonian, $\{\Phi_a \approx 0\}$ are the (first-class) primary constraints following from the definition of the canonical momenta and v^a are the associated Lagrange multipliers. From the requirement that the primary constraints should be conserved in time, we obtain in the usual way, following the Dirac algorithm [9] the secondary, tertiary, etc constraints. For later convenience we follow here a notation different from that of [10, 11], with the Latin indices a, b, c labelling the primary constraints, Greek indices α, β, γ labelling the remaining constraints, and capital Latin indices A, B, C referring to the complete set of constraints. For simplicity we shall generally refer to all constraints beyond the primary ones as 'secondary'. We denote the complete set of primary and secondary constraints by $\{\Phi_A\} = \{\Phi_a, \Phi_\alpha\}$.

Following the conjecture of Dirac [9], the generator of the gauge transformations G is given by

$$G = \sum_A \epsilon^A \Phi_A \tag{2}$$

where the gauge parameters are allowed to depend in general on time, as well as on the phase-space variables and Lagrange multipliers v^a. An infinitesimal transformation on the coordinates, generated by G, is then given by

$$\delta q^\ell = \epsilon^A [q^\ell, \Phi_A] \tag{3}$$

where a summation over repeated indices is always understood from here on.

The Poisson algebra of the constraints with themselves and with the canonical Hamiltonian, is of the form

$$[H_c, \Phi_A] = V_A{}^B \Phi_B \tag{4}$$

$$[\Phi_A, \Phi_B] = C_{AB}{}^C \Phi_C \tag{5}$$

where V_A and $C_{AB}{}^C$ may be functions of the phase-space variables.

As was shown in [6, 10, 11], G in (2) will generate a local symmetry of the corresponding total Lagrangian, provided the following relations hold:

$$\delta v^b = \frac{\mathrm{d}\epsilon^b}{\mathrm{d}t} - \epsilon^A[V_A{}^b + v^a C_{aA}{}^b] \tag{6}$$

$$0 = \frac{\mathrm{d}\epsilon^\beta}{\mathrm{d}t} - \epsilon^A[V_A{}^\beta + v^a C_{aA}{}^\beta] \tag{7}$$

where v^a are the Lagrange multipliers. In the above equations, $\mathrm{d}\epsilon^A/\mathrm{d}t$ denotes the total time derivative. For obtaining the generator of the symmetries of the original Lagrangian, only equation (7) is relevant. Equation (6) is required for consistency on the Hamiltonian level [11].

As shown in [11], the above equations can be compactly summarized in a simple differential equation for the generator G expressing its time independence

$$\frac{\partial G}{\partial t} + [G, H_T] = 0. \tag{8}$$

Equations (6) and (7) describe the restrictions imposed on the Lagrange multipliers and gauge parameters for the most general case where the structure functions depend on coordinates and momenta.

So far the solution of (7) has, in general, not been possible. Hence we shall seek a solution under two assumptions.

(a) The Poisson bracket of *any* constraint with the primary constraints is a linear combination of only the primary constraints. This implies $C_{aA}{}^\beta = 0$, and hence the absence of the last term in (7)†.
(b) The structure functions V_A^β are either constants, or may be arbitrary functions of the fields provided that there are no 'tertiary' constraints. That is, the Dirac algorithm terminates at the first level. Important examples are provided by quantum electrodynamics (QED) and quantum chromodynamics (QCD); see [11], as well as the examples discussed in the following two sections.

Though the generator of gauge transformations subject to the above assumptions has been obtained in [13], we wish to present here a more compact and transparent approach to the solution based directly on the general set of equations (7).

In order to solve equations (7) it is convenient to organize the constraints into 'families', where the parent of each family 'a' is given by a primary constraint $\phi_0^{(a)}$, and the remaining members $\phi_i^{(a)}$, $i = 1, 2 \ldots$, are recursively derived from [13]

$$[H_c, \phi_{i-1}^{(a)}] = \phi_i^{(a)} \qquad i = 1, \ldots, N_a. \tag{9}$$

The complete set of constraints is now given by $\phi_i^{(a)}$, $a = 1, \ldots, M$, $i = 0, \ldots, N_a$. Note that the subscript $i = 0$ has been reserved to label the primary constraints. With the above change

† From (8) and (2) we see that this condition also implies that

$$\frac{\partial}{\partial t} G + [G, H_c] = \mathrm{PFC} \qquad \text{(primary first class)}$$

$$[G, \mathrm{PFC}] = \mathrm{PFC}$$

which are the restrictions of [7, 13], claimed to be necessary to have a generator of Lagrangian gauge symmetries. On the other hand, it was shown in [11] that conditions (6) and (7) following from the 'master equation' (8) and the Poisson brackets (5) are sufficient for G to be the generator of local symmetries at a Lagrangian level. This shows that the conditions of [7, 13] are unnecessarily restrictive.

in notation for the constraints, the structure functions ${V_A}^B$ in (4) are correspondingly replaced by V_{ij}^{ab}, which now have the simple form

$$V_{ij}^{ab} = \delta^{ab}\delta_{i,j-1} \qquad i = 0, \ldots, N_a - 1. \tag{10}$$

In order to ensure that the constraints thus obtained are irreducible, we must adopt some systematic procedure. A possibility is to implement the Dirac algorithm level by level, descending from all primary constraints simultaneously. Scanning one by one through each member at each level, we terminate a family 'a', if at a given level N_a, the Poisson bracket of the constraint $\phi_{N_a}^{(a)}$ with H_c can be written as a linear combination of all the other constraints obtained up to this point. This ensures the irreducibility of the constraints thus obtained. Organizing the families in this particular way then implies that $V_{N_a j}^{ab} = 0$ for $j > \inf\{N_a, N_b\}$. However, whatever the procedure one adopts for obtaining the irreducible set of constraints, the Poisson bracket of the final member of each family with H_c is given by

$$[H_c, \phi_{N_a}^{(a)}] = \sum_{b=1}^{M}\sum_{j=0}^{N_b} V_{N_a j}^{ab}\phi_j^{(b)} \tag{11}$$

where M is the number of (independent) primary constraints. Correspondingly, equation (7) now reads

$$0 = \frac{\mathrm{d}\epsilon_i^{(a)}}{\mathrm{d}t} - \sum_{b=1}^{M}\sum_{j=0}^{N_b}\epsilon_j^{(b)} V_{ji}^{ba} \qquad i = 1, \ldots, N_a. \tag{12}$$

Choosing as our independent parameters those associated with the last member in each family,

$$\alpha^a := \epsilon_{N_a}^{(a)}(t) \tag{13}$$

and using (10), equations (12) take the form

$$\frac{\mathrm{d}\epsilon_i^{(a)}}{\mathrm{d}t} - \epsilon_{i-1}^{(a)} - \sum_{b=1}^{M}\alpha^b V_{N_b i}^{ba} = 0 \qquad i = 1, \ldots, N_a. \tag{14}$$

The solution to this set of equations can be constructed iteratively, by starting with the last member of a family:

$$\epsilon_{N_a-1}^{(a)} = \frac{\mathrm{d}\alpha^a}{\mathrm{d}t} - \sum_{b=1}^{M}\alpha^b V_{N_b N_a}^{ba}. \tag{15}$$

Continuing in the same fashion, one easily sees that the general solution can be written in the form

$$\epsilon_i^{(a)} = \sum_{n=0}^{N_a-i}\sum_{b=1}^{M}\frac{\mathrm{d}^n\alpha^b}{\mathrm{d}t^n}A_{i(n)}^{ba} \qquad i = 0, \ldots, N_a \tag{16}$$

with the normalization

$$A_{N_a(0)}^{ba} = \delta^{ba} \tag{17}$$

following from our choice of parametrization (13). Substituting the above ansatz into (14) and comparing powers in the time derivatives, we obtain the recursion relations

$$A_{i(n-1)}^{ba} = A_{i-1(n)}^{ba} \qquad i = 1, \ldots, N_a \tag{18}$$

Table 1.

	$n=0$	$n=1$	$n=2$	$n=3$	$n=4$	$\dots$
$i=N_a$	δ^{ba}	0	0	0	$\dots$	$\dots$
$i=N_a-1$	$-V^{ba}_{N_b,N_a}$	δ^{ba}	0	$\dots$	$\dots$	$\dots$
$i=N_a-2$	$-V^{ba}_{N_b,N_a-1}$	$-V^{ba}_{N_bN_a}$	δ^{ba}	0	$\dots$	$\dots$
$i=N_a-2$	$-V^{ba}_{N_b,N_a-2}$	$-V^{ba}_{N_bN_a-1}$	$-V^{ba}_{N_bN_a}$	δ^{ba}	0	$\dots$
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$

with the 'initial conditions'

$$A^{ba}_{i-1(0)} = -V^{ba}_{N_b i} \qquad i = 1, \dots, N_a \tag{19}$$

following from a comparison of (14) with (16). It is easy to see, that these recursion relations determine the complete solution, from which the generator of the Lagrangian gauge symmetries can be obtained. The result is summarized in table 1, where the entries are the coefficients $A^{ba}_{i(n)}$.

Using (16) in the generator (2), the infinitesimal gauge transformation (3) takes the form

$$\delta q^\ell = \sum_{b=1}^{M} \sum_{n\geqslant 0} \frac{\mathrm{d}^n \alpha^b}{\mathrm{d}t^n} \rho^\ell_{(n)b}(q, \dot{q}) \tag{20}$$

with

$$\rho^\ell_{(n)b}(q, \dot{q}) = \sum_{a=1}^{M} \sum_{j\geqslant 0} \theta(N_a - n - j) A^{ba}_{j(n)} \frac{\partial \phi^a_j}{\partial p_\ell} \tag{21}$$

where θ is the usual Heavide theta function with $\theta(0) = 1$, and where it is understood that the dependence on the canonical momenta on the right-hand side has been replaced by the respective expressions in terms of the Lagrangian variables. Expression (21) is in the form obtained by purely Lagrangian methods [2, 4].

In the case where all the families contain at most two members and the structure constants ${C_{aA}}^\beta$ vanish† (examples are provided by QED, QCD [11], and those discussed below) we can relax the above assumption concerning the constancy of the structure functions V^{ab}_{ij}, since our iterative scheme already terminates with equation (15) with $N_a = 1$ for all a, and we have for the generator

$$G = \sum_{a=1}^{M} \left[\left(\frac{\mathrm{d}\alpha^a}{\mathrm{d}t} - \sum_{b=1}^{M} \alpha^b V^{ba}_{11} \right) \phi^{(a)}_0 + \alpha^a \phi^{(a)}_1 \right]. \tag{22}$$

3. Applications

In this section we consider two examples illustrating the procedure. The first one is a modified version of the Nambu–Goto model, which illustrates the comment made in the last paragraph above concerning the absence of tertiary constraints. The second example then illustrates two ways of dealing with systems involving first- and second-class constraints.

† This will be the case if the canonical Hamiltonian can be written in the form $H_c(q, p, \xi) = H_0(q, p) + \xi^\alpha T_\alpha(q, p)$ [12], where the Lagrange multipliers ξ^α are the variables conjugate to the primary constraints, and implement via $[H_c, \phi_a] = T_a$ the secondary constraint $T_a \approx 0$ for each family.

3.1. Nambu–Goto-type model

Consider the Lagrangian [12]

$$L = \int \mathrm{d}\sigma \left(\frac{1}{2}\frac{\dot{x}^2}{\lambda} - \frac{\mu}{\lambda}\dot{x}x' + \frac{1}{2}\frac{\mu^2}{\lambda}x'^2 - \frac{1}{2}\lambda x'^2 \right) \tag{23}$$

where the 4-vector $x^\mu(\tau, \sigma)$ labels the coordinates of a 'string' parametrized by τ and σ, with the 'dot' and 'prime' denoting the derivative with respect to τ and σ, respectively. There are two primary constraints, $\pi_1 \approx 0$ and $\pi_2 \approx 0$, where π_1 and π_2 are the momenta conjugate to the fields $\lambda(\tau, \sigma)$ and $\mu(\tau, \sigma)$, respectively. Hence in our notation

$$\phi_0^{(1)} = \pi_1 \qquad \phi_0^{(2)} = \pi_2. \tag{24}$$

The canonical Hamiltonian reads

$$H_c = \int \mathrm{d}\sigma \left\{ \tfrac{1}{2}\lambda(p^2 + x'^2) + \mu p \cdot x' \right\} \tag{25}$$

where p_μ is the 4-momentum conjugate to the coordinate x^μ. The conservation in time of the primary constraints leads, respectively, to secondary constraints, which in our notation read

$$\phi_1^{(1)} = \tfrac{1}{2}(p^2 + x'^2) \approx 0 \qquad \phi_1^{(2)} = p \cdot x' \approx 0. \tag{26}$$

One readily checks that there are no further constraints.

We see that the secondary constraints are just the primary constraints of the Nambu–Goto string model. They satisfy the familiar Poisson brackets†

$$[\phi_1^{(1)}(\sigma), \phi_1^{(1)}(\sigma')] = \phi_1^{(2)}(\sigma)\partial_\sigma\delta(\sigma - \sigma') - \phi_1^{(2)}(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') \tag{27}$$

$$[\phi_1^{(1)}(\sigma), \phi_1^{(2)}(\sigma')] = \phi_1^{(1)}(\sigma)\partial_\sigma\delta(\sigma - \sigma') - \phi_1^{(1)}(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma') \tag{28}$$

$$[\phi_1^{(2)}(\sigma), \phi_1^{(2)}(\sigma')] = \phi_1^{(2)}(\sigma)\partial_\sigma\delta(\sigma - \sigma') - \phi_1^{(2)}(\sigma')\partial_{\sigma'}\delta(\sigma - \sigma'). \tag{29}$$

All other Poisson brackets vanish. The constraints are seen to be first class. In our terminology, we thus have two families, each with two members.

The canonical Hamiltonian is of the form

$$H_c = \int \mathrm{d}\sigma\, (\lambda\pi_1^{(1)}(\sigma) + \mu\phi_1^{(2)}(\sigma)). \tag{30}$$

The structure functions V_{ij}^{ab} are read off from the Poisson brackets

$$[H_c, \phi_1^{(1)}] = -\lambda\partial_\sigma\phi_1^{(2)} - 2\lambda'\phi_1^{(2)} - \mu\partial_\sigma\phi_1^{(1)} - 2\mu'\phi_1^{(1)} \tag{31}$$

$$[H_c, \phi_1^{(2)}] = -\lambda\partial_\sigma\phi_1^{(1)} - 2\lambda'\phi_1^{(1)} - \mu\partial_\sigma\phi_1^{(2)} - 2\mu'\phi_1^{(2)} \tag{32}$$

to be

$$\begin{aligned} V_{11}^{11}(\sigma, \sigma') &= V_{11}^{22}(\sigma, \sigma') = -\left(\mu(\sigma)\partial_\sigma + 2\mu'(\sigma)\right)\delta(\sigma - \sigma') \\ V_{11}^{12}(\sigma, \sigma') &= V_{11}^{21}(\sigma, \sigma') = -\left(\lambda(\sigma)\partial_\sigma + 2\lambda'(\sigma)\right)\delta(\sigma - \sigma'). \end{aligned} \tag{33}$$

Since for the example in question $N_1 = N_2 = 1$, it follows from (15), that our iterative scheme for finding the solution already ends at the first step, with $\epsilon_0^{(a)}$ given by

$$\epsilon_0^{(a)} = \frac{\mathrm{d}\alpha^a}{\mathrm{d}\tau} - \int \mathrm{d}\sigma' \sum_{b=1}^{2} \alpha^b(\sigma')V_{11}^{ba}(\sigma', \sigma). \tag{34}$$

† We suppress the τ variable.

We thus obtain

$$\epsilon_0^{(1)} = \frac{\mathrm{d}\alpha^1}{\mathrm{d}\tau} - \mu\partial_\sigma\alpha^1 + \mu'\alpha^1 - \lambda\partial_\sigma\alpha^2 + \lambda'\alpha^2 \tag{35}$$

$$\epsilon_0^{(2)} = \frac{\mathrm{d}\alpha^2}{\mathrm{d}\tau} - \mu\partial_\sigma\alpha^2 + \mu'\alpha^2 - \lambda\partial_\sigma\alpha^1 + \lambda'\alpha^1. \tag{36}$$

From (3) we now compute the corresponding transformation laws for the fields to be

$$\begin{aligned} &\delta x^\mu = \alpha^1 p^\mu + \alpha^2\partial_\sigma x^\mu \\ &\delta\lambda = \epsilon_0^{(1)} \qquad \delta\mu = \epsilon_0^{(2)}. \end{aligned} \tag{37}$$

Making use of the expressions for $\epsilon_0^{(a)}$ derived above, we verify that our results (37) agree with that quoted in the literature [12].

3.2. System with mixed constraints

As an example of a system with first- and second-class constraints consider the Abelian pure Chern–Simons theory defined by the Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\kappa\epsilon_{\mu\nu\rho}A^\mu\partial^\nu A^\rho. \tag{38}$$

The primary and secondary constraints of the system are

$$\begin{aligned} &\pi_0 \approx 0 \\ &\pi_i - \tfrac{1}{2}\kappa\epsilon_{ij}A^j \approx 0 \qquad i = 1, 2 \end{aligned} \tag{39}$$

and

$$\tfrac{1}{2}\kappa\epsilon_{ij}\partial^i A^j \approx 0 \tag{40}$$

respectively. These constraints can be grouped, respectively, into first- and second-class constraints in the following way:

$$\begin{aligned} &\Omega_0 = \pi_0 \approx 0 \\ &\Omega_3 = \partial^i\pi_i + \tfrac{1}{2}\kappa\epsilon_{ij}\partial^i A^j \approx 0 \end{aligned} \tag{41}$$

and

$$\Omega_i = \pi_i - \tfrac{1}{2}\kappa\epsilon_{ij}A^j \approx 0 \qquad i = 1, 2 \tag{42}$$

with the Poisson algebra,

$$[\Omega_i(x), \Omega_j(y)] = -\kappa\epsilon_{ij}\delta^2(x - y). \tag{43}$$

The gauge symmetries of this Lagrangian can be viewed from two points of view.

(i) The second-class constraints are implemented strongly in terms of Dirac brackets [9]

$$[A^i(x), A^j(y)] = \epsilon^{ij}\delta^2(x - y). \tag{44}$$

Following the procedure of section 2 with respect to the first-class constraints, the usual gauge symmetry $A^\mu \to A^\mu + \partial^\mu\Lambda$ of the action is then found to be generated with respect to these Dirac brackets by the first-class constraints in the particular combination

$$G = \int \mathrm{d}^2x\, (\Omega_0\partial_0\Lambda - \Omega_3\Lambda) \tag{45}$$

thus implying the usual transformation law $\delta A^\mu = \partial^\mu\Lambda$.

(ii) The mixed system is turned into a purely first-class system by following the standard embedding prescriptions [15], and the enlarged gauge symmetries are explored.

The first step consists in converting the second-class constraints (42) into first-class constraints by introducing a pair of auxiliary fields Φ^1 and Φ^2 corresponding to Ω_1 and Ω_2, respectively, and satisfying the symplectic algebra

$$[\Phi^1(x), \Phi^2(y)] = \epsilon^{ij}\delta^2(x-y). \tag{46}$$

The corresponding first-class constraints are found to be†

$$\omega_i = \pi_i - \tfrac{1}{2}\kappa\epsilon_{ij}A^j - \sqrt{\kappa}\epsilon_{ij}\Phi^j \approx 0 \qquad i = 1, 2. \tag{47}$$

One readily checks that the constraints (47) together with the constraints (41) form a first-class system. For later convenience we rewrite the constraints (41) in the form [16]

$$\begin{aligned} \omega_0 &= \Omega_0 = \pi_0 \approx 0 \\ \omega_i &= \pi_i - \tfrac{1}{2}\kappa\epsilon_{ij}A^j - \sqrt{\kappa}\epsilon_{ij}\Phi^j \approx 0 \\ \omega_3 &= \Omega_3 = \partial^i\pi_i + \tfrac{1}{2}\kappa\epsilon_{ij}\partial^i A^j \approx 0. \end{aligned} \tag{48}$$

One checks that this set of first-class constraints is, in fact, strongly involutive. For the first-class canonical Hamiltonian of the embedded system one finds [14]

$$H_c = -\kappa \int \mathrm{d}^2y\, A_0\epsilon_{ij}\partial^i\left(A^j + \frac{1}{\sqrt{\kappa}}\Phi^j\right) \tag{49}$$

satisfying the first-class algebra,

$$[H_c, \omega_0] = -\omega_0 \qquad [H_c, \omega_3] = 0 \tag{50}$$

$$[H_c, \omega_i] = 0 \qquad i = 1, 2. \tag{51}$$

From H_c we obtain the corresponding Lagrangian by following the usual procedure of writing down the corresponding phase-space path integral including the first-class constraints and integrating over the momenta conjugate to A^μ; the result is

$$\mathcal{L} = -\tfrac{1}{2}\kappa\epsilon_{\mu\nu\rho}A^\mu\partial^\rho A^\nu - \tfrac{1}{2}\epsilon_{ij}\Phi^i\partial_0\Phi^j - \sqrt{\kappa}A^0\epsilon_{ij}\partial^i\Phi^j - \sqrt{\kappa}\epsilon_{ij}\Phi^i\partial_0 A^j \tag{52}$$

or, up to a total divergence,

$$\mathcal{L} = -\tfrac{1}{2}\kappa\epsilon_{ij}\left(A^i + \frac{1}{\sqrt{\kappa}}\Phi^i\right)\partial_0\left(A^j + \frac{1}{\sqrt{\kappa}}\Phi^j\right) - \kappa A^0\epsilon_{ij}\partial^i\left(A^j + \frac{1}{\sqrt{\kappa}}\Phi^j\right). \tag{53}$$

It is easy to see that we recover from this Lagrangian the canonical Hamiltonian (49) and the full set of first-class constraints above, as well as the second-class constraints

$$\pi_{\Phi_i} = \tfrac{1}{2}\epsilon_{ij}\Phi^j. \tag{54}$$

At first it may seem that we have a clash with the first-class construction above. However, the strong implementation of these second-class constraints just leads to the symplectic algebra (46), with respect to which the first-class character of the embedded system was shown. This symplectic algebra also ensures the Lorentz covariance of the Lagrangian [17]. This demonstrates the consistency of our construction.

Let us finally examine the Lagrangian gauge symmetries implied by the first-class constraints in the embedded formulation. Because of the absence of 'tertiary' constraints, the generator of the Lagrangian gauge symmetries is found from (22) to be given by

$$G = \int \mathrm{d}^2x\, (-\dot{\epsilon}^3\omega_0 + \epsilon^3\omega_3). \tag{55}$$

† For a non-trivial example see, [14].

With $\epsilon^3 = -\Lambda$, we thus obtain for the allowed gauge transformations:

$$\begin{aligned} \delta A^0 &= \partial_0 \Lambda \\ \delta A^i &= -\epsilon^i \\ \delta \Phi^i &= \sqrt{\kappa}\,(\epsilon^i + \partial^i \Lambda). \end{aligned} \tag{56}$$

Note that in the embedded version the combination $A^i + \frac{1}{\sqrt{\kappa}}\Phi^i$ corresponds to the gauge field A^i in the original formulation, where the second-class constraints were implemented strongly. This is in accordance with the fact that

$$\delta\left(A^i + \frac{1}{\sqrt{\kappa}}\Phi^i\right) = \partial^i \Lambda \tag{57}$$

as seen from (56).

This illustrates two complementary ways of looking at the gauge symmetries of a mixed constrained system in the sense of Dirac.

4. Conclusion

To summarize, we have shown that the equations defining the restrictions to be imposed on the gauge parameters in (2) could be solved by following a simple iterative scheme, in the case where the structure functions C_{aA}^{β} in equation (7) vanish and V_A^{β} are constants. We have then applied the general ideas to the case of a non-trivial model with a two-family constraint structure, sharing some properties with the familiar Nambu–Goto model of string theory. Since each family of constraints consisted of only two members, the solution could be obtained, although the structure functions V_A^{β} were functions of the fields. We thereby recovered the local symmetry transformations quoted in the literature. All this referred to systems having only first-class constraints. We then illustrated in terms of the Abelian Chern–Simons theory, how this scheme could also be implemented for systems involving first- as well as second-class constraints.

Acknowledgment

One of the authors (RB) would like to thank the Alexander von Humboldt Foundation for providing financial support which made this collaboration possible.

References

[1] Sudarshan E C G and Mukunda N 1974 *Classical Dynamics: a Modern Perspective* (New York: Wiley)
[2] Gitman D M and Tyutin I V 1990 *Quantization of Fields with Constraints* (Heidelberg: Springer)
[3] Chaichian M and Louis Martinez D 1994 *J. Math. Phys.* **35** 6536
[4] Shirzad A 1998 *J. Phys. A: Math. Gen.* **31** 2747
[5] Costa M E V, Girotti H O and Simoes T J M 1985 *Phys. Rev.* D **32** 405
[6] Henneaux M, Teitelboim C and Zanelli J 1990 *Nucl. Phys.* B **332** 169
[7] Cabo A, Chaichian M and Louis Martinez D 1993 *J. Math. Phys.* **34** 5646
[8] Mukhanov V and Wipf A 1995 *Int. J. Mod. Phys.* A **10** 579
[9] Dirac P A M 1950 *Can. J. Math.* **2** 129
Dirac P A M 1964 *Lectures on Quantum Mechanics* Yeshiva University
[10] Banerjee R, Rothe H J and Rothe K D 1999 *Phys. Lett.* B **463** 248
[11] Banerjee R, Rothe H J and Rothe K D 1999 *Heidelberg Preprint* HD-THEP-99-24 hep-th/9907217
[12] Batlle C, Gomis J, Paris J and Roca J 1990 *Nucl. Phys.* B **329** 139

[13] Shirzad A and Shabani Moghadam M 1999 *J. Phys. A: Math. Gen.* **32** 8185
[14] Kim Y-W and Rothe K D 1998 *Int. J. Mod. Phys.* A **13** 4183
[15] Batalin I A and Fradkin E S 1986 *Phys. Lett.* B **180** 157
Batalin I A and Fradkin E S 1987 *Nucl. Phys.* **279** 514
Batalin I A and Tyutin I V 1991 *Int. J. Mod. Phys.* A **6** 3255
[16] Kim W T, Kim Y-W and Park Y-J 1999 *J. Phys. A: Math. Gen.* **32** 2461
[17] Banerjee R 1993 *Phys. Rev.* D **48** R5467